Output Stabilization via Nonlinear Luenberger Observers *

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Abstract

The present paper addresses the problem of existence of an (output) feedback law to the purposes of asymptotically steering to zero a given controlled variable, while keeping all state variables bounded, for any initial conditions in a given compact set. The problem can be viewed as an extension of the classical problem of semi-globally stabilizing the trajectories of a controlled system to a compact set. The problem also encompasses a version of the classical problem of output regulation. Assuming only the existence of a feedback law that keeps the trajectories of the zero dynamics of the controlled plant bounded, it is shown that there exists a controller solving the problem at hand. The paper is deliberately focused on theoretical results regarding the existence of such controller. Practical aspects involving the design and the implementation of the controller are left to a forthcoming work.

Keywords: Output Stabilization, Nonlinear Output Regulation, Nonlinear Observers, Lyapunov Functions, Nonminimum-phase Systems, Robust Control.

1 Introduction

The problem of controlling a system in such a way that some selected variables converge to zero while all other state variables remain bounded is a relevant problem in control theory. It includes, as special cases, the problem of asymptotic stabilization of a fixed equilibrium point and the problem of asymptotic stabilization of a fixed invariant set. It also includes design problems in which some selected variables are required to asymptotically track (or

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to asymptotically reject) certain signals generated by an independent autonomous system. Problems of this kind, usually referred to as problems of "output regulation", have been extensively studied in the past for linear systems (see [11, 19, 18]) as well as, beginning with the seminal work [24], for nonlinear systems. As a matter of fact, these problems can be viewed as problems in which a "regulated" output of an "augmented system" (a system consisting of the controlled plant and of the exogenous system generator) must be asymptotically steered to zero while all other state variables are kept bounded. As for instance pointed out in [24], the basic challenges in a problem of this type are to create an invariant set on which the desired regulated output vanishes, and to render this set asymptotically attractive.

Even though limited in scope (the design method suggested therein being only meant to secure local, and non-robust, regulation about an equilibrium point) paper [24] had the merit of highlighting a few basic concepts and ideas which shaped all subsequent developments in this area of research. These ideas include the fundamental link between the problem in question and the notion of "zero dynamics" (a concept introduced and studied earlier by the same authors), the necessity of the existence of a (controlled) invariant set on which the desired regulated output vanishes, and an embryo of design philosophy based on the idea of making this invariant set locally (and exponentially) attractive.

In the past fifteen years, the design philosophy of [24] was extended in several directions. One clear need was to move from "local" to "non-local" convergence, a goal which was pursued – for instance – in [26], [22] and [31], [6], where different approaches (at increasing levels of generality) have been proposed. Another concern was to obtain design methods which are insensitive, or even robust, with respect to model uncertainties. This issue was originally addressed in [21], where it was shown how, under appropriate hypotheses, the property of (local) asymptotic regulation can be made robust with respect to plant parameter variations, extending in this way a celebrated property of linear regulators.

In the presence of plant parameter variations, the challenge is to design a (parameter independent) controller in such a way that the closed-loop system possesses a (possibly parameter *dependent*) attractive invariant set on which the regulated output vanishes. The two issues of forcing the existence of such an invariant set and of making the latter (locally or non-locally) attractive are of course interlaced and this is precisely what, in the past years, has determined the various scenarios under which different solutions to the problem have been proposed. In the paper [21], for instance, this was achieved by assuming that the set all feed-forward controls which force the regulated output to be identically zero had to be generated by a single (parameter-independent) *linear* system. This assumption was weakened in [13], in [12] and, subsequently, in [7], where it was replaced by the assumption that the controls in question are generated by a single (parameter-independent) *nonlinear* system, uniformly observable in the sense of [20].

The crucial observation that made the advances in [7] and [12] possible was the realization that the two issues of forcing the existence of an invariant set (on which the regulated variable vanishes) and of making the latter attractive are intimately related to, and actually can be cast as, the problem of designing a (nonlinear) *observer*. As a matter of fact, the

design method suggested in [7] was based almost entirely on the construction of a nonlinear "high-gain" observer following the methods of Gauthier-Kupka [20], while the design method suggested in [12] was based almost entirely on the construction of a nonlinear adaptive observer following the methods of Bastin-Gevers [3] and Marino-Tomei [25].

Having realized that the design of observers is instrumental in the design of controllers which solve the problem in question, the idea came to examine whether alternative options, in the design of observers, could be of some help in weakening the assumptions even further. This turns out to be the case, as shown in the present paper, if the approach to the design of nonlinear observers outlined by Kazantis-Kravaris ([28]) and then pursued by Kresselmeier-Engel ([29]), by Krener-Xiao ([27]) and by Andrieu-Praly ([1]) is adopted.

While in all earlier contributions it was assumed that the controls which force the regulated output to be identically zero could be interpreted as outputs of a (in general nonlinear) system having special observability properties (which eventually became part of the controller), a crucial property highlighted in the proof of Theorem 3 of [1] shows that no assumption of this kind is actually needed. The controls in question can *always* be generated by means of a system of appropriate dimension whose dynamics are linear but whose output map is a nonlinear (and, in general, only continuous but not necessarily locally Lipschitzian) map. Once this system is embedded in the controller, boundedness of all closed-loop trajectories and convergence to the desired invariant set can be guaranteed, as in the earlier contributions [7] and [12], by a somewhat standard paradigm which blends practical stabilization with a small-gain property for feedback interconnection of systems which are input-to-state stable (with restrictions).

The purpose of this paper is to provide a complete proof of how the results of [1] can be exploited for the design of a controller solving the problem and also to show how some technical hypotheses used in the asymptotic analysis of [8] can be totally removed, yielding in this way a general theory cast only on a very simple and meaningful assumption. This paper is deliberately meant to present only all the theoretical results needed to show the *existence* of the solution of the problem in question. Issues related to practical aspects involving constructive design and implementation will be dealt with in a forthcoming work.

The paper is organized as follows. In the next section the main framework under which the problem is solved is presented and discussed. Then Section 3 presents an outline of the main results concerning the existence of the output feedback regulator. Technical proofs of the results in this section are postponed to Appendices A and B. Section 4 concludes the paper with some with final remarks.

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and, for $\mathcal{C}$ a closed subset of $\mathbb{R}^n$, $|x|_{\mathcal{C}} = \min_{y \in \mathcal{C}} |x - y|$ denotes the distance of x from $\mathcal{C}$. For $\mathcal{S}$ a subset of $\mathbb{R}^n$, cl$\mathcal{S}$ and int$\mathcal{S}$ are the closure of $\mathcal{S}$ and the interior of $\mathcal{S}$ respectively, and $\partial\mathcal{S}$ its boundary. For the smooth dynamical system $\dot{x} = f(x)$, the value at time t of the solution passing through x_0 at time $t = 0$ will be written as $x(t, x_0)$. Somewhere the more compact notation $x(t)$ will be used instead of $x(t, x_0)$, when the initial condition is clear from the context. A set $\mathcal{S}$ is said to be locally forward (backward) invariant for $\dot{x} = f(x)$ if there is a time $t_0 > 0$ ($-t_0 < 0$) such

that, for each $x_0 \in \mathcal{S}$, $x(t,x_0) \in \mathcal{S}$ for all $t \in [0,t_0)$ ($t \in (-t_0, 0]$). The set is locally invariant if it is locally backward and forward invariant. The set is (forward/backward) invariant if it is locally (forward/backward) invariant with $t_0 = \infty$. For a locally Lipschitz function $V(t)$ we define the Dini's derivative of V at t as

$$D^+V(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h) - V(t)]\,.$$

By extension, when $V(t)$ is obtained by evaluating V along a solution $x(t,x_0)$, we denote also

$$D^+V(x_0) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(x(h,x_0)) - V(x_0)]\,. \tag{1}$$

Note that if $\lim\sup = \lim$, this is simply $L_f V(x_0)$, the Lie derivative at x_0 of V along f.

2 The framework

2.1 The problem of output stabilization and the main result

We consider in what follows a nonlinear smooth system described by

$$\begin{array}{rcl} \dot z &=& f(z,\zeta) \\ \dot\zeta &=& q(z,\zeta) + u \end{array} \tag{2}$$

with state $(z,\zeta) \in \mathbb{R}^n \times \mathbb{R}$ and *control* input $u \in \mathbb{R}$, and with unknown initial conditions $(z(0),\zeta(0))$ ranging in a known arbitrary compact set $Z \times \Xi \subset \mathbb{R}^n \times \mathbb{R}$.

Associated with (2) there is a *controlled* output $e \in \mathbb{R}$ expressed as

$$e = h(z,\zeta) \tag{3}$$

and a *measured* output $y \in \mathbb{R}^p$ expressed as

$$y = k(z,\zeta) \tag{4}$$

in which $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^p$ are smooth functions.

For system (2)-(3)-(4) the problem of semiglobal (with respect to $Z \times \Xi$) output stabilization is defined as follows. Find, if possible, an output feedback controller of the form

$$\begin{array}{rcl} \dot\eta &=& \varphi(\eta,y) \\ u &=& \varrho(\eta,y) \end{array} \tag{5}$$

with state $\eta \in \mathbb{R}^\nu$ and a compact set $M \subset \mathbb{R}^\nu$ such that, in the associated closed loop system

$$\begin{array}{rcl} \dot z &=& f(z,\zeta) \\ \dot\zeta &=& q(z,\zeta) + \varrho(\eta,k(z,\zeta)) \\ \dot\eta &=& \varphi(\eta,k(z,\zeta)) \\ e &=& h(z,\zeta)\,, \end{array} \tag{6}$$

the positive orbit of $Z \times \Xi \times M$ is bounded and, for each $(z(0), \zeta(0), \eta(0)) \in Z \times \Xi \times M$,

$$\lim_{t\to\infty} e(t) = 0\,.$$

The problem at issue will be solved under the following main assumption which, roughly speaking, requires that system

$$\dot z \;=\; f(z,\zeta) \tag{7}$$

viewed as a system with input ζ and output (4), is "stabilizable" by output feedback in an appropriate sense. In more precise terms the assumption in question is formulated as follows.

Assumption. There exists a compact set $\mathcal{A} \subset \mathbb{R}^n$, a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$ and a smooth map $\Phi : \mathbb{R}^p \to \mathbb{R}$ such that:

($\mathbf{a}_1$) the set $\mathcal{A}$ is locally asymptotically stable for the system

$$\dot z \;=\; f(z,\alpha(z)) \tag{8}$$

with a domain of attraction $\mathcal{D} \supset Z$;

($\mathbf{a}_2$) $h(z,\alpha(z)) = 0$ for all $z \in \mathcal{A}$.

($\mathbf{a}_3$) $\Phi(k(z,\zeta)) = \zeta - \alpha(z)$ for all $(z,\zeta) \in \mathbb{R}^n \times \mathbb{R}$.

Comments to this assumption are postponed after the next theorem which presents the main result of the paper.

Theorem 1 *There exists an $m > 0$, a controllable pair $(F,G) \in \mathbb{R}^{m\times m}\times\mathbb{R}^{m\times 1}$, a continuous function $\gamma : \mathbb{R}^m \to \mathbb{R}$ and, for any compact set $M \subset \mathbb{R}^m$, a continuous function $\kappa : \mathbb{R}^p \to \mathbb{R}$, such that the controller*

$$\begin{array}{rcll} \dot\eta &=& F\eta + Gu & \qquad \eta(0) \in M \\ u &=& \gamma(\,\eta\,) + v & \\ v &=& \kappa(y) & \end{array} \tag{9}$$

solves the problem of semiglobal (with respect to $Z \times \Xi$) output stabilization.

Remark. It is worth noting that the previous Assumption could be weakened by asking not for a static but rather for a dynamic output feedback stabilizer. More precisely for all the results presented in the paper to hold, it suffices to assume the existence of a smooth system

$$\begin{array}{rcl} \dot\xi &=& \sigma(\xi, u_\xi) \\ y_\xi &=& \beta(\xi, u_\xi) \end{array}$$

with initial state allowed to range on a compact subset Σ of $\mathbb{R}^l$, of a pair of smooth functions $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\Phi : \mathbb{R}^p \to \mathbb{R}$, and of a compact set $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^l$ such that:

($\mathbf{a}_1'$) the set $\mathcal{A}$ is locally asymptotically stable for the system

$$\begin{aligned} \dot z &= f(z, \beta(\xi, \alpha(z))) \\ \dot \xi &= \sigma(\xi, \alpha(z)) \,, \end{aligned}$$

with a domain of attraction $\mathcal{D} \subset Z \times \Sigma$,

($\mathbf{a}_2'$) $h(z, \beta(\xi, \alpha(z))) = 0$ for all $(z, \xi) \in \mathcal{A}$,

and condition ($\mathbf{a}_3$) above holds. This, however, is not deliberately pursued in what follows, as it would only add unnecessary complications, without any extra conceptual value. $\triangleleft$

2.2 Remarks on the framework and the result

Since a crucial requirement in the problem of semiglobal output stabilization is boundedness of the closed-loop trajectories, it seems natural to postulate the existence of a "virtual" control $\zeta(\cdot)$ that keeps the trajectories of (7) bounded. Assumption ($\mathbf{a_1}$) does this by asking that the control in question be a state feedback law, namely a control $\zeta = \alpha(z)$ under which the trajectories of (7) asymptotically approach a compact set $\mathcal{A}$. If this is the case, the change of variables

$$\chi = \zeta - \alpha(z)$$

changes system (2) into a system of the form

$$\begin{aligned} \dot z &= f(z, \alpha(z) + \chi) \\ \dot \chi &= \tilde q(z, \chi) + u \end{aligned} \tag{10}$$

in which

$$\tilde q(z, \chi) = q(z, \alpha(z) + \chi) + \frac{\partial \alpha}{\partial z} f(z, \alpha(z) + \chi).$$

Accordingly, the controlled output e becomes

$$e = h(z, \alpha(z) + \chi)\,.$$

Clearly, to take advantage of the fact that the trajectories of (8) with initial conditions in Z, as required in part ($\mathbf{a_1}$) of the Assumption, are attracted by a compact set $\mathcal{A}$, it would be desirable to have asymptotically χ converging to zero.

Of course since the same "virtual" control appears in the map which expresses the controlled variable e, and the latter is required to asymptotically decay to 0, it is also appropriate to assume, as done in ($\mathbf{a}_2$), that $h(z, \alpha(z))$ vanishes on $\mathcal{A}$. If this were to occur, in fact, then also the controlled variable e would converge to zero and the problem would be solved.

To make χ converging to zero, one might wish to appeal to (somewhat standard) "high-gain" arguments and have $u = -k\chi$, which would be an admissible control law because, as required in part ($\mathbf{a_3}$) of the Assumption, $\chi = \Phi(y)$ is available from the measured output. However, it is well known (see e.g. [34]) that to have χ asymptotically converging to zero

in a "high-gain" scheme, it is somewhat necessary that the "coupling" term $\tilde{q}(z,\chi)$ between the upper and the lower subsystem of (10) asymptotically vanishes. More specifically, it is necessary that $\tilde{q}(z,0)$ be vanishing on the set $\mathcal{A}$ to which the state z of the upper subsystem converges if ζ decays to zero. Now, in general, there is no guarantee that $\tilde{q}(z,0)$ would vanish on $\mathcal{A}$ and this is why a more elaborate controller has to be synthesized. As a matter of fact, the main result of the paper is that a suitable *dynamic* controller makes it sure that a property of this kind is achieved.

A special case covered by the previous setup is the one in which system (7) can be given the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1) \\ \dot{z}_2 &= f_2(z_1, z_2, \zeta)\,. \end{aligned} \tag{11}$$

In this case, it is clear that the dynamics of z_1 is a totally *autonomous* dynamics, which can be viewed as an "exogenous" signal generator. This is the way in which the classical problem of output regulation is usually cast. Depending on the control scenario, the variable z_1 may assume different meanings. It may represent exogenous disturbances to be rejected and/or references to be tracked. It may also contain a set of (constant or time-varying) uncertain parameters affecting the controlled plant.

In this context, it is important to note that the proposed framework encompasses a number of problems which have been recently addressed (see, among others, [26], [32], [12], [6], [4], [13]) and rely upon various versions of the so-called "minimum-phase" property. More specifically all the aforementioned works consider systems having relative degree $r \geq 1$ and normal form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x}_1 &= f(w, x_1, Cx_2) \\ \dot{x}_2 &= Ax_2 + B\zeta \\ \dot{\zeta} &= q(w, x_1, x_2, \zeta) + u \end{aligned} \tag{12}$$

in which $w \in \mathbb{R}^s$ represents an exogenous input, $x_1 \in \mathbb{R}^\ell$, $x_2 \in \mathbb{R}^{r-1}$, $\zeta \in \mathbb{R}$ and A, B, C is a triplet in "prime" form, with controlled and measured output respectively given by

$$\begin{aligned} e &= Cx_2 \\ y &= \mathrm{col}(x_2, \zeta)\,. \end{aligned}$$

Note that $\mathrm{col}(x_2, \zeta) = \mathrm{col}(e, \dot{e}, \ldots, e^{r-1})$. [1]

For this class of systems one of the main assumptions under which the problem of output regulation has been solved is the one requiring that the "zero dynamics"

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x}_1 &= f(w, x_1, 0) \end{aligned} \tag{13}$$

[1] The case in which $y = \mathrm{col}(x_2, \zeta)$ is known in the literature at the case of *partial state feedback*, to emphasize the fact that not only the error Cx_2 is available as measured output but also all its first $r-1$ time derivatives. This is not a restriction, though, since – as shown for instance in [15] and [34] – so long as convergence from a compact set of initial conditions is sought, all components of y can always estimated by means of an "approximate" observer driven only by its first component Cx_2.

satisfy some stability requirement. For instance in [32] the assumption in question asks for the existence of a differentiable map $\pi : \mathbb{R}^s \to \mathbb{R}^\ell$ whose graph $\mathcal{A}' = \{(w, x_1) \in \mathbb{R}^s \times \mathbb{R}^\ell \ : \ x_1 = \pi(w)\}$ is invariant and locally exponentially stable for (13), uniformly with respect to the exogenous variable w, with a domain of attraction containing the assigned compact set of initial conditions. This assumption has been substantially weakened in the recent work [6] (see also [7] and [4]) by asking a form of " weak minimum-phase" property, in which the compact set $\mathcal{A}'$ above is rather the graph of a *set-valued* map.[2] Under this assumption, as shown in [4] (see also [12]), it is possible to argue the existence of a matrix K, designed via high-gain arguments, such that the system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{x}_1 & = & f(w, x_1, Cx_2) \\ \dot{x}_2 & = & Ax_2 + BKx_2 \end{array} \tag{14}$$

possesses an *invariant* compact set $\mathcal{A} = \{(w, x_1, x_2) \ : \ (w, x_1) \in \mathcal{A}', x_2 = 0\}$ which is locally exponentially stable for (14) with a domain of attraction containing the set of initial conditions.

It is clear that the case described above perfectly fits in the framework presented in Section 2.1, with the (w, x_1, x_2) subsystem in (12) playing the role of the z-subsystem in (2), with the controlled output and measured output map in (3)-(4) respectively equal to Cx_2 and (x_2, ζ), and with assumptions $(\mathbf{a_1})$-$(\mathbf{a_2})$-$(\mathbf{a_3})$ automatically satisfied with $\mathcal{A} = \{(w, x_1, x_2) \ : \ (w, x_1) \in \mathcal{A}', x_2 = 0\}$, $\alpha(z) = Kx_2$ and $\Phi(y) = (\,-K \quad 1\,)\, y$.

On the other hand, the framework set up in Section 2.1 makes it possible to deal with some form of non-minimum-phase systems, extending in this way the class of systems which can be treated.

3 Main results

3.1 The basic approach

In this section we overview the main steps which will be followed to prove Theorem 1. Technical proofs of the results given here are presented in Appendix B.

We consider the closed-loop system (2), (9) which, after the change of coordinates

$$\zeta \to \chi = \zeta - \alpha(z) \qquad\qquad \eta \to x = \eta - G\chi\,,$$

can be rewritten as

$$\begin{array}{rcl} \dot{z} & = & f_0(z) + f_1(z, \chi) \\ \dot{x} & = & Fx - Gq_0(z) - Gq_1(z, \chi) + FG\chi \\ \dot{\chi} & = & q_0(z) + q_1(z, \chi) + \gamma(x + G\chi) + v \end{array} \tag{15}$$

[2]As shown in [6] this assumption is a straightforward consequence of the boundedness of the trajectories of (13).

in which

$$
\begin{aligned}
f_0(z) &:= f(z,\alpha(z)) \\
q_0(z) &:= q(z,\alpha(z)) - \frac{\partial\alpha(z)}{\partial z} f(z,\alpha(z))
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
f_1(z,\chi) &:= f(z,\alpha(z)+\chi) - f(z,\alpha(z)) \\
q_1(z,\chi) &:= q(z,\alpha(z)+\chi) - q(z,\alpha(z)) - \frac{\partial\alpha(z)}{\partial z}\left[f(z,\alpha(z)+\chi) - f(z,\alpha(z))\right].
\end{aligned}
$$

Observe that we have $f_1(z,0) \equiv 0$ and $q_1(z,0) \equiv 0$ for all $z \in \mathbb{R}^n$.

In what follows, system (15) is seen as a system with input v, output χ (which is, according to its definition and to assumption ($\mathbf{a}_3$), a nonlinear function of the original output y) and initial conditions contained in a set of the form $Z \times X \times C$ in which $X \subset \mathbb{R}^m$ and $C \subset \mathbb{R}$ are compact sets dependent on Ξ and M. A controller of the form (9) solves the problem at issue if, for some map $\hat{\kappa} : \mathbb{R}^n \to \mathbb{R}^n$, the control law $v = \hat{\kappa}(\chi)$ is such that all trajectories of (15) originating from $Z \times X \times C$ are bounded and

$$
\lim_{t\to\infty} \chi(t) = 0\,, \qquad \lim_{t\to\infty} |z(t))|_{\mathcal{A}} = 0\,. \tag{17}
$$

As a matter of fact, since systems (2) -(9) and (15) are diffeomorphic, boundedness of the trajectories of (15) with initial conditions in $Z\times X\times C$ implies boundedness of the trajectories of (2) -(9) originating from $Z \times \Xi \times M$. Furthermore, by virtue of assumption ($\mathbf{a_2}$), since $h(\cdot)$ is a continuous function, condition (17) implies also that $\lim_{t\to\infty} e(t) = 0$, namely that the problem of semiglobal output stabilization is solved.[3]

By virtue of this fact, in the following we focus our attention on system (15) and we prove that (15) controlled by $v = \hat{\kappa}(\chi)$ has bounded trajectories and (17) hold. To this end, for notational convenience, denote

$$
p = \mathrm{col}(z, x)
$$

and rewrite system (15) in the more compact form

$$
\begin{aligned}
\dot{p} &= M(p) + N(p,\chi) \\
\dot{\chi} &= H(p) + K(p,\chi) + v
\end{aligned}
\tag{18}
$$

in which $M(\cdot)$ and $H(\cdot)$ are defined as

$$
M(p) = \begin{pmatrix} f_0(z) \\ Fx - Gq_0(z) \end{pmatrix} \tag{19}
$$

and

$$
H(p) = q_0(z) + \gamma(x) \tag{20}
$$

[3]Note that the map $\kappa(\cdot)$ to be used in (9) is actually $\hat{\kappa}(\Phi(\cdot))$.

and $N(\cdot)$ and $K(\cdot)$ are therefore suitable remainder functions which satisfy necessarily $N(p,0) = 0$ and $K(p,0) = 0$ for all p. Consistently set $P = Z \times X$ so that the initial conditions of (18) range in $P \times C$. System (18) is recognized to be a system in normal form with relative degree one (with respect to the input v and output χ) and zero dynamics given by

$$\dot{p} = M(p)\,. \tag{21}$$

Thus, following consolidated knowledge about stabilization of *minimum-phase* nonlinear systems (see [9], [2], [34]), the capability of stabilizing (18) by output feedback is expected to strongly rely upon asymptotic properties of the zero dynamics (21). This is confirmed by the next two results showing that the existence of an asymptotically stable attractor for system (21) is sufficient to achieve boundedness of trajectories and *practical* stabilization (Theorem 2), which becomes *asymptotic* if the function $H(\cdot)$ vanishes on the attractor (Theorem 3). These results, which in the context of this paper represent building blocks for proving Theorem 1, are interesting by their own as they represent an extension of well-known stabilization paradigms for systems with equilibria (see [34]) to the case of systems of the form (18), with zero dynamics (21) possessing compact attractors.

Theorem 2 *Consider system (18) with $M(\cdot)$ at least locally Lipschitz function and $N(\cdot)$, $H(\cdot)$, $K(\cdot)$ at least continuous functions. Let the initial conditions be in $P \times C$. Assume that system (21) has a compact attractor $\mathcal{B}$ which is asymptotically stable with a domain of attraction $\mathcal{D} \supset P$. Then for all $\epsilon > 0$ there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the trajectories of (18) with $v = -\kappa\chi$ are bounded and $\limsup_{t\to\infty} |\chi(t)| \leq \epsilon$ and $\limsup_{t\to\infty} |p(t)|_{\mathcal{B}} \leq \epsilon$.*

Theorem 3 *In addition to the hypotheses of the previous theorem, assume that $H(p)|_{\mathcal{B}} = 0$. Then there exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that the trajectories of (18) with $v = \kappa(\chi)$ are bounded and $\lim_{t\to\infty} \chi(t) = 0$ and $\lim_{t\to\infty} |p(t)|_{\mathcal{B}} = 0$. If, additionally, $H(\cdot)$ and $K(\cdot)$ are locally Lipschitz and the set $\mathcal{B}$ is also locally exponentially stable for (21), then there exists $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the same properties hold with $v = -\kappa\chi$.*

For the proofs of these theorems the reader is referred to the sections B.1 and B.2 respectively.

Motivated by these results (and in particular by Theorem 3), we turn our attention on the study of the zero dynamics (21) (with $M(\cdot)$ as in (19)) and on the function $H(\cdot)$ in (20), by looking for the existence of a controller which guarantees the basic requirements behind Theorem 3, with in particular $H(p)|_{\mathcal{B}} = 0$. Details in this direction are presented in the next subsection.

3.2 The properties of the "core subsystem" (21)

The crucial result which will be proved in this part is that, under the assumption presented in section 2.1, there is a choice of the pair (F, G) and of the map $\gamma(\cdot)$ which guarantee the existence of an asymptotically stable compact attractor $\mathcal{B}$ for (21), on which the function

$H(\cdot)$ in (20) vanishes. Moreover the projection of $\mathcal{B}$ on the z coordinates coincides with $\mathcal{A}$. In view of the arguments discussed in the previous subsection, this, along with an appropriate choice of $\kappa(\cdot)$ whose existence is claimed in Theorem 3, substantially proves Theorem 1.

The result in question is proven in the next three propositions. To this end, note that the core subsystem (21) in the original coordinates (z, x) is expressed as

$$\begin{aligned} \dot{z} &= f_0(z) \\ \dot{x} &= Fx - Gq_0(z) \end{aligned} \tag{22}$$

with initial condition in $Z \times X$. The first proposition is related to the first basic requirement behind Theorem 2, namely the existence of a locally asymptotically stable attractor for (22).

More precisely, under the only requirement that F is an Hurwitz matrix, it is shown the existence of a set which is *forward invariant* and *locally asymptotically stable* for (22). The set in question is described by the graph of a map.

Proposition 1 *Consider system (22) with $\alpha(\cdot)$ such that condition ($\mathbf{a}_1$) holds and let (F, G) be any pair with F Hurwitz. Then:*

(i) there exists at least one continuous map $\tau : \mathbb{R}^n \to \mathbb{R}^m$ such that the set

$$\mathrm{graph}(\tau|_{\mathcal{A}}) := \{(z, x) \in \mathcal{A} \times \mathbb{R}^m \ : \ x = \tau(z)\} \tag{23}$$

is forward invariant for (22).

(ii) The set $\mathrm{graph}(\tau|_{\mathcal{A}})$ is locally asymptotically stable for (22) with a domain of attraction containing $Z \times X$. Furthermore the set in question is also locally exponentially stable for (22) if $\mathcal{A}$ is such for (8).

The proof of this proposition can be found in Section B.3.

Remark. Indeed, there might be many different continuous maps τ having the property (i) of proposition 1. However, it turns out that if $\mathcal{A}_0$ is any compact *subset* of $\mathcal{A}$ which is *invariant* for (8), then for each $z \in \mathcal{A}_0$ there is one and only one $x_z \in \mathbb{R}^m$ such that the set $\cup_{z \in \mathcal{A}_0}\{(z, x_z)\}$ is invariant for (22). In particular,

$$x_z = -\int_{-\infty}^{0} e^{-Fs} G q_0(z(s, z)) ds$$

where $z(s, z)$ denotes the value at time $t = s$ of the solution of $\dot{z} = f_0(z)$ passing through $z \in \mathcal{A}_0$ at time $t = 0$ (see [5]). $\triangleleft$

The second crucial requirement imposed by Theorem 3 is that the function $H(\cdot)$ in (20) vanishes on the asymptotically stable attractor $\mathrm{graph}(\tau|_{\mathcal{A}})$. Here is where the precise choice of the pair (F, G) and of the map $\gamma(\cdot)$ play a role. In particular note that, by definition of $H(\cdot)$ in (20) and of $\mathrm{graph}(\tau|_{\mathcal{A}})$ in (23), it turns out that

$$H(p)|_{\mathrm{graph}(\tau|_{\mathcal{A}})} = (q_0(z) + \gamma \circ \tau(z))|_{\mathcal{A}} \tag{24}$$

from which it is apparent that $\gamma(\cdot)$ should be chosen to satisfy $\gamma \circ \tau(z) = -q_0(z)$ for all z in $\mathcal{A}$. It is easy to realize that the possibility of choosing $\gamma(\cdot)$ in this way is intimately related to the fact that the map τ satisfies the *partial (with respect to $q_0(\cdot)$) injectivity condition*

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \text{for all } z_1, z_2 \in \mathcal{A}\,. \tag{25}$$

As τ is dependent on the pair (F, G), the next natural point to be addressed is if there exists a choice of (F, G) yielding the desired property for $\tau(\cdot)$. To this end is devoted the next proposition which claims that, indeed, there exists a suitable choice of (F, G), with F Hurwitz, such that the associated map $\tau(\cdot)$ satisfies the required partial injectivity condition. Besides others technical constraints on the choice of F which will be better detailed in the proof of the Proposition 2, the main requirement on F is given by its dimension which is required to be sufficiently large with respect to the dimension of z.

Proposition 2 *Set*

$$m = 2 + 2n\,.$$

Then there exist a controllable pair $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times 1}$, with F a Hurwitz matrix, and a class-$\mathcal{K}$ function $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|q_0(z_1) - q_0(z_2)| \le \varrho(|\tau(z_1) - \tau(z_2)|) \qquad \textit{for all } z_1, z_2 \in \mathcal{A} \tag{26}$$

in which $\tau(\cdot)$ is a map (associated to F) with the properties indicated in Proposition 1.

For the proof of this proposition the reader is referred to Section B.4.

Remark. Going through the proof of the previous proposition, it turns out that the pair (F, G) be chosen as any $(2n+2)$-dimensional real representation of the $(n+1)$-dimensional complex pair (F_c, G_c), with $F_c = \mathrm{diag}(\lambda_1, \ldots, \lambda_{r+1})$, $G_c = (g_1, \ldots, g_{r+1})^{\mathrm{T}}$ in which g_i are arbitrary not zero real numbers and λ_i are $n+1$ complex numbers taken arbitrarily outside a set of zero Lebesgue measure and with real part smaller than ℓ, a real number related to the Lipschitz constant of $f_0(\cdot)$ (see Proposition 4). $\triangleleft$

It turns out that the injectivity property (26) is a sufficient condition for the map $\gamma(\cdot)$ to exist. This is formalized in the next final proposition, proved in Section B.5, which states that if (26) holds then there exists a map $\gamma(\cdot)$ which makes $H(\cdot)$ vanishing on the attractor graph$(\tau|_{\mathcal{A}})$. The map $\gamma(\cdot)$ can be claimed, in general, to be only continuous. It is also Lipschitz in the special case in which the class-$\mathcal{K}$ function $\varrho(\cdot)$ in (26) is such.

Proposition 3 *Let $\tau(\cdot)$ be a continuous map satisfying (26) with $\mathcal{A}$ a closed set. Then there exist a continuous map $\gamma : \mathbb{R}^m \to \mathbb{R}$ such that*

$$q_0(z) + \gamma \circ \tau(z) = 0 \qquad \forall\, z \in \mathcal{A}\,. \tag{27}$$

If, in addition, the function $\varrho(\cdot)$ in (26) is linearly bounded at the origin, then the map γ is Lipschitz.

Combining the results of all the previous propositions, it appears that it is sufficient to choose the pair (F, G) of suitable dimension (with F Hurwitz) according to Proposition 2 and to choose $\gamma(\cdot)$ in order to satisfy relation (27). In fact, so doing, we are guaranteed that the compact set $\mathcal{B} = \text{graph}(\tau|_{\mathcal{A}})$ is locally asymptotically stable for (22) with the map (20) which is vanishing on $\mathcal{B}$. This, indeed, makes it possible to apply Theorem 3 and to conclude the existence of a continuous function $\kappa(\cdot)$, completing in this way the synthesis of the controller.

Remark. The reader who is familiar with recent developments of the theory of nonlinear state observers will find interesting to compare the previous results with the design method proposed by Kazantzis and Kravaris in [28] and pursued in [29], [27] and [1]. In the framework of [28], system (22) can be identified with the cascade of an "observed" system $\dot{z} = f_0(z)$ with output $y_z = q_0(z)$ driving an "observer" $\dot{x} = Fx - Gy_z$. If the map $\tau(\cdot)$ has a left inverse $\tau_\ell^{-1}(\cdot)$, the observer in question provides a state estimate $\hat{z} = \tau_\ell^{-1}(x)$. Such a left-inverse, as shown in [1], always exists provided that the dimension of x is sufficiently large, if the pair (f_0, q_0) has appropriate *observability* properties. In the present context of output stabilization, though, left invertibility of $\tau(\cdot)$ is not needed. In fact, what the controller is expected to do is only the reproduction of the output $q_0(z(t))$ and not of the full state $z(t)$ of the "observed system". This motivates the absence of observability hypotheses on the pair (f_0, q_0). $\triangleleft$

4 Conclusions

This paper is focused on the existence of an output feedback law that asymptotically steers to zero a given controlled variable, while keeping all state variables bounded, for any initial conditions in a fixed compact set. The proposed framework encompasses and extends a number of existing results in the fields of output feedback stabilization and output regulation of nonlinear systems. The main assumption under which the theory is developed is the existence of a state feedback control law able to achieve boundedness of the trajectories of the zero dynamics of the controlled plant. In this sense the result presented here is applicable for a wide class of non-minimum-phase nonlinear systems not tractable in existing frameworks. In the paper only results regarding the existence of the controller solving the problem at hand have been presented while practical aspects involving its design and implementation are left to a forthcoming work.

A Converse Lyapunov result

Consider a system of the form

$$\dot{p} = f(p) \qquad p \in \mathbb{R}^n\,, \tag{28}$$

in which $f(p)$ is a C^k (with k sufficiently large) function, with initial condition ranging over a fixed *compact* set P. For system (28) assume the existence of a compact set $\mathcal{B} \subset \mathbb{R}^n$ which

is forward invariant and asymptotically stable for (28), with a domain of attraction $\mathcal{D} \supset P$. More precisely, by setting

$$|p|_{\mathcal{B}/\mathcal{D}} = \left(1 + \frac{1}{|p|_{\partial \operatorname{cl} \mathcal{D}}}\right) |p|_{\mathcal{B}}\,,$$

we assume that the set $\mathcal{B}$ satisfies the following two properties:
Uniform stability: there exists a class $\mathcal{K}$ function φ such that for any $\alpha > 0$

$$|p_0|_{\mathcal{B}/\mathcal{D}} \leq \alpha \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \leq \varphi(\alpha) \quad \forall\, t \geq 0\,;$$

Uniform attractivity: there exists a continuous function $T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\alpha > 0$ and $\epsilon > 0$

$$|p_0|_{\mathcal{B}/\mathcal{D}} \leq \alpha \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \leq \epsilon \quad \forall\, t \geq T(\alpha, \epsilon)\,.$$

We say that $\mathcal{B}$ is also *locally exponentially stable* for (28) if there exist $M \geq 1$, $\lambda > 0$ and $c_0 > 0$ such that

$$|p_0|_{\mathcal{B}/\mathcal{D}} \leq c_0 \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \leq M e^{-\lambda t} |p_0|_{\mathcal{B}/\mathcal{D}}\,.$$

In this framework it is possible to formulate the following converse Lyapunov result which claims the existence of a *locally Lipschitz* Lyapunov function vanishing on the attractor. The result is not formally proved as it can be easily deduced by the arguments presented in [35] (see in particular Theorem 22.5 and the related Theorems 22.1 and 19.2 in the quoted reference).

Theorem 4 *Under the above uniform stability and uniform attractivity conditions, there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ with the following properties:*

(a) there exist class $\mathcal{K}_\infty$ functions $\underline{a}(\cdot)$, $\overline{a}(\cdot)$ such that

$$\underline{a}(|p|_{\mathcal{B}/\mathcal{D}}) \leq V(p) \leq \overline{a}(|p|_{\mathcal{B}/\mathcal{D}}) \qquad \forall\, p \in \mathcal{D}\,;$$

(b) there exists $c > 0$ such that

$$D^+ V(p) \leq -c V(p) \qquad \forall\, p \in \mathcal{D}\,;$$

(c) for all $\alpha > 0$ there exists $L_\alpha > 0$ such that for all p_1, $p_2 \in \mathcal{D}$ such that $|p_1|_{\mathcal{B}/\mathcal{D}} \leq \alpha$, $|p_2|_{\mathcal{B}/\mathcal{D}} \leq \alpha$ the following holds

$$|V(p_1) - V(p_2)| \leq L_\alpha |p_1 - p_2|\,.$$

If $\mathcal{B}$ is also locally exponentially stable for (28) then property (a) holds with $\underline{a}(\cdot)$, $\overline{a}(\cdot)$ linear near the origin.

With this result at hand, it is also possible to formulate a local Input-to-State Stability result for system (28) forced by an external signal. This is formalized in the next lemma.

Lemma 1 *Let $x : \mathbb{R}_+ \to \mathbb{R}^m$ be a C^0 function. Consider the system*

$$\dot{p} = f(p) + \ell(p, x(t)) \tag{29}$$

in which $p \in \mathbb{R}^n$ and $\ell(p, 0) = 0$ for all $p \in \mathbb{R}^n$. The functions $f(\cdot), \ell(\cdot)$ are C^1. Suppose that system (28) satisfies the assumptions expressed before. Then there exist functions $\beta(\cdot, \cdot)$ and $\gamma(\cdot)$, respectively of class $\mathcal{KL}$ and $\mathcal{K}$, and a $d^* > 0$ such that if

$$|p_0|_{\mathcal{B}} \le d^* \qquad \textit{and} \qquad |x(t)| \le d^* \quad \textit{for all } t \ge 0 \tag{30}$$

then the right maximal interval of definition of $p(t, p_0)$ is $[0, +\infty)$ and we have

$$|p(t, p_0)|_{\mathcal{B}} \le \max\left\{ \beta(|p_0|_{\mathcal{B}}, t)\,,\, \gamma(\max_{\tau \in [0,t]} |x(\tau)|) \right\} \qquad \textit{for all } t \ge 0. \tag{31}$$

If the set $\mathcal{B}$ is also locally exponentially stable for (28) then there exist $N > 1$, $k > 0$ and $\bar{\gamma} > 0$ such that (31) modifies as

$$|p(t, p_0)|_{\mathcal{B}} \le N e^{-kt} |p_0|_{\mathcal{B}} + \bar{\gamma} \max_{\tau \in [0,t]} |x(\tau)| \qquad \textit{for all } t \ge 0. \tag{32}$$

Proof. Pick $\beta > 0$ such that if $|p|_{\mathcal{B}} \le \beta$ then $p \in \mathcal{D}$ and note that there exists $d_\beta > 1$ such that for all p satisfying $|p|_{\mathcal{B}} \le \beta$ then

$$|p|_{\mathcal{B}/\mathcal{D}} \le d_\beta |p|_{\mathcal{B}}. \tag{33}$$

As $\ell(\cdot)$ is differentiable and $\ell(p, 0) = 0$, there is an $\hat{\ell} > 0$ such that for all $|p|_{\mathcal{B}} \le \beta$ and $|x| \le d^*$

$$|\ell(p, x)| \le \hat{\ell} |x|\,.$$

So consider the Lyapunov function V given by Theorem 4. By using properties (b) and (c) of this theorem setting $\alpha = d_\beta \beta$, we obtain for the system (29), so long as $|p_1|_{\mathcal{B}} < \beta$ and $|x| \le d^*$,

$$\begin{aligned}
D^+V(p_1, x) &= \limsup_{h \to 0^+} \frac{1}{h} [V(p(h, p_1)) - V(p_1)] \\
&= \limsup_{h \to 0^+} \frac{1}{h} [V(p_1 + hf(p_1) + h\ell(p, x)) - V(p_1)] \\
&\le \limsup_{h \to 0^+} \frac{1}{h} [V(p_1 + hf(p_1) + h\ell(p_1, x)) - V(p_1 + hf(p_1))] \\
&\qquad\qquad + \limsup_{h \to 0^+} \frac{1}{h} [V(p_1 + hf(p_1)) - V(p_1)] \\
&\le \limsup_{h \to 0^+} \frac{1}{h} L_\alpha h \ell(p_1, x) - cV(p_1) \le L_\alpha \hat{\ell} |x| - cV(p_1)\,.
\end{aligned} \tag{34}$$

Now assume (30) holds. Let $p(t, p_0)$, shortly rewritten $p(t)$, be the corresponding solution of (29). Let $[0, T_0)$ be its right maximal interval of definition when restricted to take values in the open set $\{p : |p|_{\mathcal{B}} < \beta\}$. (34) holds for $p_1 = p(t)$ and all t in $[0, T_0)$. This implies

$$V(p(t)) \leq e^{-c(t-t_0)} V(p_0) + \frac{L_\alpha \hat{\ell}}{c} \max_{\tau \in [0,t]} |x(\tau)| \qquad \forall t \in [0, T_0)\,. \tag{35}$$

This, in view of property (a) in Theorem 4, yields

$$|p(t)|_{\mathcal{B}} \leq |p(t)|_{\mathcal{B}/\mathcal{D}} \leq \underline{a}^{-1}(2e^{-ct}\,\overline{a}(|p_0|_{\mathcal{B}/\mathcal{D}})) + \underline{a}^{-1}(2\frac{L_\alpha \hat{\ell}}{c} \max_{\tau \in [0,t]} |x(\tau)|) \qquad \forall t \in [0, T_0)\,. \tag{36}$$

By using (33), it follows that if d^* is chosen so that

$$d^\star \leq \min\{\frac{c}{2L_\alpha \hat{\ell}}\,\underline{a}(\beta/3)\,,\, \frac{1}{d_\beta}\,\overline{a}^{-1}(\frac{1}{2}\,\underline{a}(\beta/3))\}$$

we have

$$|p(t)|_{\mathcal{B}} < \beta \qquad \forall t \in [0, T_0)\,.$$

From the definition of T_0, it must be infinite. So we have established that (36) holds for all $t \geq 0$ if (30) is satisfied. This proves the first part of the result. The second part of the result, namely that under exponential stability the bound (32) holds, follows immediately by (36) by using the fact that the functions $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ can be linear near the origin. ◁

B Proofs

B.1 Proof of Theorem 2

Proof. The study of the feedback interconnection (18) can be done by means of arguments which are quite similar to those used in [23] to prove some of the main stabilization results of [34]. In doing this, we take advantage of the converse Theorem 4 presented in Appendix A.

Let $V : \mathcal{D} \to \mathbb{R}$ be the function given by Theorem 4. Pick a number $a > 0$ such that $C \subset B_a := \{\chi \in \mathbb{R} \ :\ |\chi| \leq a\}$ and $P \subset V^{-1}([0, a])$ (which is possible because of property (a) in Theorem 4). Define

$$\hat{c} = \max_{(p,\chi) \in V^{-1}([0,a+1]) \times B_{a+1}} |H(p) + K(p, \chi)|\,.$$

Also, since $N(p, \chi)$ is differentiable and vanishes at $\chi = 0$, there is a number $\hat{n}$ such that

$$|N(p, \chi)| \leq \hat{n}|\chi| \qquad \forall (p, \chi) \in V^{-1}([0, a + 1]) \times B_{a+1}.$$

Finally, by property (c) in Theorem 4, there is a number L_V such that

$$|V(p_1) - V(p_2)| \leq L_V |p_1 - p_2| \qquad \forall (p_1, p_2) \in V^{-1}([0, a+1])^2.$$

Then, by choosing $v = -k\chi$ in the χ-dynamics in (18), we get (see notation (1))

$$D^+|\chi| \leq -\kappa|\chi| + \hat{c} \qquad \forall (p, \chi) \in V^{-1}([0, a+1]) \times B_{a+1} \tag{37}$$

Also, by following the same lines as in (34), we get

$$D^+V(p) \leq L_V \hat{n}|\chi| - cV(p) \qquad \forall (p, \chi) \in V^{-1}([0, a+1]) \times B_{a+1}.. \tag{38}$$

So now consider a solution $(p(t), \chi(t))$ issued from a point in $P \times C \subset V^{-1}([0, a]) \times B_a$. Let $[0, T_1)$ be its right maximal interval of definition when restricted to take values in the open set $\mathrm{int}(V^{-1}([0, a+1]) \times B_{a+1})$. It follows that both (37) and (38) hold for $(p(t), \chi(t))$ when t is in $[0, T_1)$. They give successively, for all t in $[0, T_1)$

$$\begin{aligned}
|\chi(t)| &\leq e^{-\kappa t}\, a + \frac{\hat{c}}{\kappa}\left(1 - e^{-\kappa t}\right) \\
V(p(t)) &\leq e^{-ct}\, a + L_V \hat{n} \left(\frac{\hat{c}}{\kappa}\frac{1 - e^{-ct}}{c} + \frac{e^{-ct} - e^{-\kappa t}}{\kappa - c}\left[a - \frac{\hat{c}}{\kappa}\right]\right) \\
&\leq a + L_V \hat{n}\left(\frac{\hat{c}}{c\kappa} + \frac{a}{\kappa - c}\right)
\end{aligned}$$

Hence, by selecting κ to satisfy

$$\kappa > \max\left\{2\hat{c}, (c + 3aL_V\hat{n}), \frac{3L_V\hat{n}\hat{c}}{c}\right\},$$

we get, for all t in $[0, T_1)$

$$|\chi(t)| \leq a + \frac{1}{2} \qquad \text{and} \qquad V(p(t)) \leq a + \frac{2}{3}$$

This says that the solution remains in $V^{-1}([0, a + \frac{2}{3}]) \times B_{a+\frac{1}{2}}$. So from its definition, T_1 is infinite. Then, from (37), we get

$$\limsup_{t \to +\infty} |\chi(t)| \leq \frac{\hat{c}}{\kappa}$$

With (38), this in turn implies

$$\limsup_{t \to +\infty} V(p(t)) \leq \frac{L_V \hat{n}\hat{c}}{\kappa}.$$

In view of property (a) in Theorem 4, the latter yields

$$\limsup_{t \to +\infty} |p(t)|_{\mathcal{B}} \leq \underline{a}^{-1}\left(\frac{L_V\hat{n}\hat{c}}{\kappa}\right).$$

So the conclusion of Theorem 2 holds if we further impose to κ to satisfy

$$\kappa > \max\left\{\frac{\hat{c}}{\epsilon}, \frac{L_V\hat{n}\hat{c}}{\underline{a}(\epsilon)}\right\}$$

◁

B.2 Proof of Theorem 3

Proof. The proof of this result follows by standard small gain arguments. Let $\kappa(\chi) = -\alpha(\chi)$ where $\alpha(\cdot)$ is a continuous function such that $\alpha(0) = 0$ and $\chi\alpha(\chi) > 0\,\forall\,\chi \neq 0$. By mimicking the proof of Theorem 2 it is possible to show that for any $\epsilon > 0$ there exist a $\kappa^\star > 0$ and a $T > 0$ such that, if $|\alpha(|\chi|)| \geq \kappa^\star|\chi|$ then each trajectory of the closed-loop system issued from the compact set $P \times C$ satisfies

$$|p(t)|_{\mathcal{B}} \leq 2\epsilon \qquad \text{and} \qquad |\chi(t)| \leq 2\epsilon \qquad \text{for all } t \geq T \qquad .$$

Observe that Lemma 1 applies to the p-component of the closed loop solution. So let $d^\star$ be given by this lemma. By picking ϵ above satisfying $2\epsilon \leq d^\star$ and by applying Lemma 1, (after time T,) we obtain

$$|p(t)|_{\mathcal{B}} \leq \max\left\{\beta(|p(T)|_{\mathcal{B}}, t-T)\,,\, \gamma(\max_{\tau\in[T,t]}|\chi(\tau)|)\right\} \qquad \text{for all } t \geq T\,. \tag{39}$$

With the properties of the functions H and K, there exist of class-$\mathcal{K}$ functions $\varrho_h(\cdot)$ and $\varrho_q(\cdot)$ such that

$$|H(p)| \leq \varrho_h(|p|_{\mathcal{B}}) \qquad |K(p,\chi)| \leq \varrho_k(|\chi|)\,.$$

Clearly $\varrho_h(\cdot)$ and $\varrho_k(\cdot)$ can be taken linearly bounded at the origin if $H(\cdot)$ and $K(\cdot)$ are locally Lipschitz. We obtain, for all (p,χ),

$$D^+|\chi| \;\leq\; \varrho_h(|p|_{\mathcal{B}}) + \varrho_k(|\chi|) - |\alpha(\chi)|\,.$$

So let us choose $\alpha(\cdot)$ so that

$$|\alpha(\chi)| \geq 3\max\{\varrho_h(\bar{\gamma}^{-1}(|\chi|)),\, \varrho_k(|\chi|),\, \kappa^\star|\chi|\} + |\chi|$$

where $\bar{\gamma}(\cdot)$ is a class $\mathcal{K}$ function such that $\bar{\gamma}\circ\gamma(s) < s$ for all $s \in \mathbb{R}^+$ with γ given by Lemma 1 (see (31)). This gives

$$D^+|\chi| \;\leq\; -|\chi| + \left[\varrho_h(|p|_{\mathcal{B}}) - \varrho_h(\bar{\gamma}^{-1}(|\chi|))\right]\,.$$

So for the closed loop solution, we get

$$|\chi(t)| \;\leq\; \max\left\{\exp(-(t-T))|\chi(T)|\,,\, \sup_{s\in[T,t)}\bar{\gamma}(|p(s)|_{\mathcal{B}})\right\}$$

for all $t \geq T$. From this and (39) the first claim of the Theorem follows by small gain arguments. The second claim of the theorem immediately follows by the previous arguments and by (32) in Lemma 1.$\triangleleft$

B.3 Proof of Proposition 1

Let $\mathcal{O}(Z)$ denote the positive orbit of Z under the flow of

$$\dot{z} = f_0(z)\,, \tag{40}$$

which is a bounded and forward invariant set for (40), such that $\mathcal{A} \subset \mathcal{O}(Z)$. Moreover let $\hat{\mathcal{O}}(Z)$ be a compact strict superset of $\mathcal{O}(Z)$ such that $\hat{\mathcal{O}}(Z) \subset \mathcal{D}$ and define the system

$$\dot{\hat{z}} = a_0(\hat{z}) f_0(\hat{z}) \tag{41}$$

in which $a(\hat{z}) : \mathbb{R}^n \to \mathbb{R}$ is any bounded smooth function such that

$$a_0(\hat{z}) = \begin{cases} 1 & \hat{z} \in \mathcal{O}(Z) \\ 0 & \hat{z} \in \mathbb{R}^n \setminus \hat{\mathcal{O}}(Z)\,. \end{cases}$$

Let $\hat{z}(t, z_0)$ and $z(t, z_0)$ denote the flows of (41) and, respectively, (40) and note that, as a consequence of the fact that $\mathcal{O}(Z)$ is forward invariant and that systems (40) and (41) agree on $\mathcal{O}(Z)$, it turns out that

$$\hat{z}(t, z_0) = z(t, z_0) \qquad \text{for all } z_0 \in \mathcal{O}(Z) \text{ and } t \geq 0\,. \tag{42}$$

Moreover note that for any $\hat{z}_0 \in \mathbb{R}^n$, (41) has a unique solution $\hat{z}(t, \hat{z}_0)$ which is defined and bounded on $t \in (-\infty, \infty)$.

Define now

$$\begin{array}{rccl} \tau & : & \mathbb{R}^n & \to & \mathbb{R}^m \\ & & z & \mapsto & \displaystyle\int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s, z)) ds \end{array} \tag{43}$$

which, as a consequence of the fact that F is Hurwitz and $q_0(\hat{z}(s, z))$ is bounded and continuous in z for any $s \in \mathbb{R}$, is a well-defined *continuous* map. We show now that $\text{graph}(\tau|_{\mathcal{A}}) = \{(z, \xi) \in \mathcal{A} \times \mathbb{R}^m \ : \ x = \tau(z)\}$ is a forward invariant set for (22). For, pick $z_0 \in \mathcal{A}$ and $x_0 \in \mathbb{R}^m$, let $(z(t, z_0), x(t, z_0, x_0))$ denote the value at time t of the solution of (22) passing through (z_0, x_0) at time $t = 0$, and note that for all $t \geq 0$ (using (42))

$$\begin{aligned} x(t, z_0, \tau(z_0)) &= e^{Ft}\tau(z_0) + \int_0^t e^{F(t-s)} G q_0(z(s, z_0)) ds \\ &= e^{Ft} \int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s, z_0)) ds + \int_0^t e^{F(t-s)} G q_0(z(s, z_0)) ds \\ &= \int_{-\infty}^{t} e^{F(t-s)} G q_0(\hat{z}(s, z_0)) ds = \int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s + t, z_0)) ds \\ &= \tau(\hat{z}(t, z_0)) = \tau(z(t, z_0))\,. \end{aligned} \tag{44}$$

This, along with the fact that $\mathcal{A}$ is forward invariant for (40) and subset of $\mathcal{O}(Z)$, proves that $\text{graph}(\tau|_{\mathcal{A}})$ is *forward invariant* for (22).

We prove now item (ii) of the proposition. To this end note that, by (44), it follows that

$$L_{a_0 f_0}\tau(z) = F\tau(z) - Gq_0(z)$$

for all $z \in \mathcal{A}$. Defining $\tilde{x} := x - \tau(z)$, the previous relation yields that $\dot{\tilde{x}}(t) = F\tilde{x}(t)$ for all $t \geq 0$ and for all initial states $x_0 \in \mathbb{R}^m$ and $z_0 \in \mathcal{A}$. This, the fact that F is Hurwitz and that $\mathcal{A}$ is locally asymptotically (exponentially) stable for (40) yield immediately the desired result. $\triangleleft$

B.4 Proof of Proposition 2

The result will be proved by taking the "complex" pair

$$F = \mathrm{diag}(\lambda_1, \ldots, \lambda_{r+1}) \qquad G = (g, \ldots, g)^{\mathrm{T}} \tag{45}$$

in which $\lambda_i \in \mathbb{C}_\ell = \{\lambda \in \mathbb{C} \;:\; \mathrm{Re}\lambda < -\ell\}$, $i = 1, \ldots, r+1$, $\ell > 0$, and $g \neq 0$. Once proved the result for the $(r+1)$-dimensional pair in (45), the claim of the proposition follows by taking any $(2r+2)$-dimensional "real" representation of (45).

By bearing in mind the definition of the map τ in (43) note that, as a consequence of the choice of F and G in (45), it turns out that

$$\tau(z) = \begin{pmatrix} \tau_{\lambda_1}(z) & \tau_{\lambda_2}(z) & \cdots & \tau_{\lambda_{r+1}}(z) \end{pmatrix}^{\mathrm{T}} \qquad \tau_{\lambda_i}(z) = \int_{-\infty}^{0} e^{-\lambda_i s}\, g\, q_0(\hat{z}(s,z))ds\,. \tag{46}$$

We will prove next that there exists an $\ell > 0$ such that by arbitrarily choosing λ_i, $i = 1, \ldots, r+1$, in $\mathbb{C}_\ell \setminus S$, where S is a set of zero Lebesgue measure, then the map τ is such that

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \forall\, z_1, z_2 \in \mathbb{R}^n\,. \tag{47}$$

More precisely we will prove that, having defined

$$\Upsilon = \{(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; q_0(z_1) \neq q_0(z_2)\}\,,$$

the set

$$S = \{(\lambda_1, \ldots, \lambda_{r+1}) \in \mathbb{C}_\ell^{r+1} \quad : \quad \exists\, (z_1, z_2) \in \Upsilon \;:\; \tau_{\lambda_i}(z_1) = \tau_{\lambda_i}(z_2) \quad \forall\, i = 1, \ldots, r+1\} \tag{48}$$

has zero Lebesgue measure in $\mathbb{C}^{r+1}$ for a proper choice of ℓ. To this end the following Theorem, proved in a more general setting in [1] (see also ([10])), plays a crucial role.

Theorem 5 *Let Ω and Υ be open subsets of $\mathbb{C}$ and $\mathbb{R}^{2n}$ respectively. Let $(\varpi, \lambda) \in \Upsilon \times \Omega \mapsto \delta_\tau(\varpi, \lambda) \in \mathbb{C}$ be a function which is holomorphic in λ for each $\varpi \in \Upsilon$ and C^1 for each $\lambda \in \Omega$. If, for each pair $(\varpi, \lambda) \in \Upsilon \times \Omega$ for which $\delta_\tau(\varpi, \lambda)$ is zero there exists an integer k satisfying*

$$\begin{aligned} &\frac{\partial^i \delta_\tau}{\partial \lambda^i}(\varpi, \lambda) = 0 \qquad \textit{for all } i \in \{0, \ldots, k-1\} \\ &\frac{\partial^k \delta_\tau}{\partial \lambda^k}(\varpi, \lambda) \neq 0 \end{aligned} \tag{49}$$

then the set

$$S = \bigcup_{\varpi \in \Upsilon} \{(\lambda_1, \ldots, \lambda_{r+1}) \in \Omega^{r+1} \quad : \quad \delta_\tau(\varpi, \lambda_1) = \ldots = \delta_\tau(\varpi, \lambda_{r+1}) = 0\}$$

has zero Lebesgue measure in $\mathbb{C}^{r+1}$.

To apply this theorem to our context we first observe the following.

Proposition 4 *There exists an* $\ell > 0$ *such that for all* $\lambda_i \in \mathbb{C}_\ell$, $i = 1, \ldots, r+1$, *the map* $\tau(\cdot)$ *in (46) is* C^1.

Proof. The map $\tau(\cdot)$ in (46) is C^1 if the functions $e^{-\lambda_i s}\, g\, \partial q_0(\hat{z}(s,z))/\partial z$, $i = 1, \ldots, r+1$, are integrable on $s \in (-\infty, 0]$ for all $z \in \mathbb{R}^n$ (see [17]). Consider the expansion

$$\frac{\partial q_0(\hat{z}(s,z))}{\partial z} = \Big[\frac{\partial q_0}{\partial z}\Big]_{[z=\hat{z}(s,z)]} \frac{\partial \hat{z}(s,z)}{\partial z}\,.$$

By definition, there is a number M such that $|\hat{z}(s,z)| \le M$ for all $s \le 0$ and all $z \in \mathbb{R}^n$. This, along with the fact that $q_0(z)$ is C^1, shows that the first factor is bounded on $(-\infty, 0] \times \mathbb{R}^n$. As for the second factor, bearing in mind the notation introduced in Section B.3, observe that

$$\frac{d}{ds}\frac{\partial \hat{z}(s,z)}{\partial z} = \Big[\frac{\partial a_0(z) f_0(z)}{\partial z}\Big]_{[z=\hat{z}(s,z)]} \frac{\partial \hat{z}(s,z)}{\partial z}$$

Letting

$$\bar{f} = \max_{z \in \mathbb{R}^n} \frac{\partial a_0(z) f_0(z)}{\partial z}$$

we obtain

$$|\frac{\partial \hat{z}(s,z)}{\partial z}| \le e^{\bar{f}|s|}$$

for all s and for all $z \in \mathbb{R}^n$. From this, the result immediately follows with $\ell = \bar{f}$. $\triangleleft$ *(End proof Proposition 4)*

Now set $\varpi := (z_1, z_2)$ and

$$\delta_\tau(\varpi, \lambda) = \int_{-\infty}^{0} e^{-\lambda s}\, g\, [q_0(\hat{z}(s,z_1)) - q_0(\hat{z}(s,z_2))]\, ds = \tau_\lambda(z_1) - \tau_\lambda(z_2)\,.$$

This function is C^1 in $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ and it is holomorphic in $\lambda \in \mathbb{C}_\ell$ for every $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ (see [30], Chap. 19, p. 367). Moreover, as

$$\int_{-\infty}^{0} e^{-as}\, |g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))\,|^2\, ds < +\infty$$

for all $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ and for all $a < 0$, the Plancherel Theorem can be invoked to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\delta_\tau(\varpi, a+is)|^2 ds = \int_{-\infty}^{0} e^{-2as}\left|g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))\right|^2 ds \tag{50}$$

for all $a < 0$ and for all $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$.

Now note that, for $\varpi = (z_1, z_2) \in \Upsilon$, we have $q_0(z_1) \neq q_0(z_2)$ and by continuity of flow with respect to time, there exists a time $t_1 < 0$ such that

$$|g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))| > 0 \qquad \text{for all } s \in (t_1, 0]$$

which, combined with (50), yields

$$\int_{-\infty}^{\infty} |\delta_\tau(\varpi, a+is)|^2 ds > 0\,.$$

This implies that, for each $\varpi \in \Upsilon$, the function $\lambda \mapsto \delta_\tau(\varpi, \lambda)$ is not identically zero on $\mathbb{C}_\ell$. Since it is holomorphic, it turns out that for each $(\varpi, \lambda) \in \Upsilon \times \mathbb{C}_\ell$ it is possible to find an integer k satisfying (49). Hence Theorem 5 can be applied to obtain the desired result, namely that the set (48) has zero Lebesgue measure.

By this result we are guaranteed that by arbitrarily picking $r+1$ complex eigenvalues in $\mathbb{C}_\ell \setminus S$ (with ℓ dictated by Proposition 4) of F defined in (45), condition (47) is satisfied. From this it is easy to show that there exists a class-$\mathcal{K}$ function satisfying (26). For, define

$$\varphi(s) = \sup_{\substack{|\tau(z_1) - \tau(z_2)| \leq s \\ z_1, z_2 \in \mathcal{A}}} |q_0(z_1) - q_0(z_2)|\,.$$

This function is increasing and, as a consequence of (47), $\varphi(0) = 0$. Moreover it is possible to prove that $\varphi(s)$ is continuous at $s = 0$. For, suppose that it is not, namely, as $\varphi(\cdot)$ is increasing and $\varphi(0) = 0$, suppose that there exists a $\varphi^\star > 0$ such that $\lim_{s \to 0^+} \varphi(s) = \varphi^\star$. This implies that there exist sequences $\{z_{1n}\}$, $\{z_{2n}\}$ in $\mathcal{A}$, such that $|q_0(z_{1n}) - q_0(z_{2n})| \geq \varphi^\star/2$ and $|\tau(z_{1n}) - \tau(z_{2n})| < 1/n$ for any $n \in \mathbb{N}$. But, as $\mathcal{A}$ is bounded, there are subsequences of $\{z_{1n}\}$, $\{z_{2n}\}$ which, for $n \to \infty$, converge to $z_1^\star$, $z_2^\star$ respectively. As $\tau(\cdot)$ and $q_0(\cdot)$ are continuous $\tau(z_1^\star) - \tau(z_2^\star) = 0$ and $|q_0(z_1^\star) - q_0(z_2^\star)| \geq \varphi^\star/2$ which contradict (47). Hence $\varphi(s)$ is continuous at $s = 0$. With this result at hand, define the candidate class-$\mathcal{K}$ function

$$\varrho(s) = \frac{1}{s}\int_{s}^{2s} \varphi(\sigma) d\sigma \; + \; s$$

which satisfies

$$\varphi(s) \leq \varrho(s)\,. \tag{51}$$

By construction this function is continuous for all $s > 0$ and, as $\varphi(s)$ is continuous at $s = 0$ and by (51), it is also continuous at $s = 0$. Moreover, by (51) and by definition of $\varphi(\cdot)$, (26) is also satisfied.

B.5 Proof of Proposition 3

By the result of the previous proposition we know that

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \forall\, z_1, z_2 \in \mathcal{A}\,.$$

For any $x \in \tau(\mathcal{A})$, let $[x] = \{z \in \mathcal{A} : \tau(z) = x\}$. The previous property shows that the map $q_0(\cdot)$ is constant on $[x]$. As a consequence, there is a well defined function $\gamma_0 : \tau(\mathcal{A}) \to \mathbb{R}$ such that

$$\gamma_0(\tau(z)) = -q_0(z), \qquad \forall z \in \mathcal{A}\,.$$

In fact, the value $\gamma_0(x)$ at any $x \in \tau(\mathcal{A})$ is simply defined by taking any $z \in [x]$ and setting $\gamma_0(x) := -q_0(z)$. Moreover, by (26), the map in question is also continuous. Now note that, as $\mathcal{A}$ is compact and $\tau(\cdot)$ and $q_0(\cdot)$ are continuous maps, $\tau(\mathcal{A}) \subset \mathbb{R}^m$ and $q_0(\mathcal{A}) \subset \mathbb{R}$ are compact sets. From this the Tietze's extension Theorem (see, for instance, Theorem VII.5.1 in [14]) can be invoked to claim the existence of a continuous map $\gamma : \mathbb{R}^m \to \mathbb{R}$ which agrees with γ_0 on $\tau(\mathcal{A})$. This implies that $q_0(z) + \gamma \circ \tau(z) = 0$ for all $z \in \mathcal{A}$ and proves the first claim of the proposition.

Furthermore if $\varrho(\cdot)$ is linearly bounded at the origin, by compactness arguments it is possible to claim the existence of a positive $\bar{\varrho}$ such that $|q_0(z_1) - q_0(z_2)| \leq \bar{\varrho}|\tau(z_1) - \tau(z_2)|$. It follows that γ_0 is a Lipschitz function on $\tau(\mathcal{A})$. From this the Kirszbraun Theorem (see, for instance, Theorem 2.10.43 in [16]) yields the existence of a Lipschitz map $\gamma : \mathbb{R}^m \to \mathbb{R}$, with Lipschitz constant ϱ, which agrees with γ_0 on $\tau(\mathcal{A})$. This completes the proof of the proposition.

References

[1] Andrieu V. and Praly L., On the existence of a Kazantzis-Kravaris observer, 43rd IEEE Conference on Decision and Control, December 14-17, Bahamas.

[2] A. Bacciotti, Linear feedback: the local and potentially global stabilization of cascade systems, Proceedings of the 2nd IFAC Symp. Nonlinear Contr. Syst. Design, 1992, Bordeaux, France.

[3] G. Bastin and M. R. Gevers, Stable adaptive observers for non-linear time varying systems, *IEEE Trans. Autom. Contr.*, **AC-33**: 650–657, 1988.

[4] C.I. Byrnes, A. Isidori, L. Marconi, Further results on output regulation by pure error feedback, 16th IFAC World Congress, Prague, July 3-8, 2005.

[5] C.I. Byrnes and A. Isidori, Exploring the Notion of Steady-State Response of a Nonlinear System: Ideas, Tools and Applications, accepted for publication on *IEEE Control Systems Magazine*.

[6] C.I. Byrnes, A. Isidori, Limit sets, zero dynamics and internal models in the problem of nonlinear output regulation, *IEEE Trans. on Automatic Control*, **AC-48**, pp. 1712–1723, 2003.

[7] C.I. Byrnes, A. Isidori, Nonlinear Internal Models for Output Regulation, *IEEE Trans. on Automatic Control*, **AC-49**, pp. 2244-2247, (2004).

[8] C.I. Byrnes, A. Isidori, L. Praly, On the Asymptotic Properties of a System Arising in Non-equilibrium Theory of Output Regulation, Preprint of the Mittag-Leffler Institute, Stockholm, 18, 2002-2003, spring.

[9] C.I. Byrnes and A. Isidori Asymptotic stabilization of minimum-phase nonlinear systems. *IEEE Trans. on Automatic Control*, **AC-36**, pp. 1122-1137, 1991.

[10] J.M. Coron, On the stabilization of controllable and observable systems by output feedback law, Mathematics of Control, Signals and Systems, 7:187-216, 1994

[11] E.J. Davison, The robust control of a servomechanism problem for linear time-invariant multivariable systems, *IEEE Trans. Autom. Contr.*, **AC-21**: 25–34, 1976.

[12] F. Delli Priscoli, L. Marconi and A. Isidori, A New Approach to Adaptive Nonlinear Regulation, ArXiv report, Reference OC/0404511, 2004.

[13] Z. Chen and J. Huang, Global robust servomechanism problem of lower triangular systems in the general case, *Systems and Control Letters*, **52**: pp. 209–220, 2004.

[14] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[15] F. Esfandiari F, HK. Khalil, Output feedback stabilization of fully linearizable systems. *International Journal of Control* **56**: pp. 1007–1037, 1992.

[16] H. Federer, Geometric measure theory, Springer Verlag 1969 (reprint in 1996).

[17] W. Fleming, *Functions of Several Variables*, Springer Verlag, 1977.

[18] B.A. Francis, The linear multivariable regulator problem, *SIAM J. Contr. Optimiz.*, **14**: 486–505, 1977.

[19] B.A. Francis and W. M. Wonham. The internal model principle of control theory. *Automatica*, **12**: 457–465, 1976.

[20] J.P. Gauthier and I. Kupka, *Deterministic Observation Theory and Applications*, Cambridge University Press, 2001.

[21] J. Huang and C.F. Lin. On a robust nonlinear multivariable servomechanism problem. *IEEE Trans. Autom. Contr.*, **AC-39**: 1510–1513, 1994.

[22] A. Isidori, A remark on the problem of semiglobal nonlinear output regulation, *IEEE Trans. on Automatic Control*, **AC-42**: 1734-1738, 1997.

[23] A. Isidori. *Nonlinear Control Systems II.* Springer Verlag (New York, NY), 1st edition, 1999.

[24] A. Isidori and C.I. Byrnes. Output regulation of nonlinear systems. *IEEE Trans. Autom. Contr.*, **AC-25**: 131–140, 1990.

[25] R. Marino and P. Tomei, Global adaptive observers for nonlinear systems via filtered transformations, *IEEE Trans. on Automatic Control*, **AC-37**, pp. 1239–1245, 1992.

[26] H. Khalil, Robust servomechanism output feedback controllers for feedback linearizable systems, *Automatica*, **30**: 1587–1599, 1994.

[27] A.J.Krener and M.Xiao, Nonlinear observer design in the Siegel domain, *SIAM J. Contr. and Optmiz.*, **41**, pp. 932–953, 2002 and **43**, pp. 377–378, 2004.

[28] Kazantzis K. and C. Kravaris, Nonlinear observer design usign Lyapunov's auxiliary theorem. Systems and Control Letters, 34, pp. 241-247, 1998.

[29] G. Kreisselmeier and R. Engel, Nonlinear observer for autonomous Lipschitz continuous systems, *IEEE Trans. Autom. Contr.*, **AC 48** 451–464, 2003.

[30] Rudin, Real and complex analysis, McGraw-Hill, Inc. 1966.

[31] A. Serrani, A. Isidori and L. Marconi, Semiglobal output regulation for minimum-phase systems, *International Journal on Robust and Nonlinear Control*, **10**, pp. 379–396, 2000.

[32] A. Serrani, A. Isidori, L. Marconi, Semiglobal nonlinear output regulation with adaptive internal model, *IEEE Trans. Autom. Contr.*, **AC-46**: 1178-1194, 2001.

[33] J. Szarski, *Differential Inequalities*, Polska Akademia Nauk (Warszawa), 1967.

[34] A.R. Teel and L. Praly, Tools for semiglobal stabilization by partial state and output feedback. *SIAM J. Control Optim.*, **33**, pp. 1443–1485, 1995.

[35] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer Verlag (New York, NY), 1975.